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# Bounded Quasi-Interpolatory Polynomial Operators

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We construct bounded polynomial operators, similar to the classical de la Vallée Poussin operators in the theory of Fourier series, which preserve polynomials of a certain degree, but are defined in terms of the values of the function rather than its Fourier coefficients. © 1999 Academic Press

*Key Words:* orthogonal polynomials; weighted approximation; de la Vallée Poussin means.

## 1. INTRODUCTION

Let $f$ be a continuous $2\pi$-periodic function, and for integer $m \geqslant 1$, $s_m^*(f)$ be the $m$th partial sum of its trigonometric Fourier series. It is well known (see, e.g., [10]) that the de la Vallée Poussin operators

$$v_n^*(f) := \frac{1}{n} \sum_{k=n+1}^{2n} s_k^*(f), \qquad n = 1, 2, ...,$$

are linear operators with the following interesting properties, where for integer $n \geqslant 0$, $\mathbb{H}_n$ denotes the class of all trigonometric polynomials of order not exceeding $n$. For all integer $n \geqslant 1$, and continuous $2\pi$-periodic functions $f$,

$$v_n^*(f) \in \mathbb{H}_{2n-1}, \tag{1.1}$$

$$v_n^*(T) = T, \qquad T \in \mathbb{H}_n, \tag{1.2}$$



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and

$$\max_{-\pi\leqslant x\leqslant\pi} |v_n^*(f, x)| \leqslant 6 \max_{-\pi\leqslant x\leqslant\pi} |f(x)|. \tag{1.3}$$

These operators are naturally of great importance in the study of trigonometric polynomial approximation. Namely, it follows immediately from (1.2) that

$$\max_{-\pi\leqslant x\leqslant\pi} |f(x)-v_n^*(f, x)| \leqslant 7E_n(f),$$

where $E_n(f)$ denotes the best approximation in the sup-norm by trigonometric polynomials of degree at most $n$. Let us finally mention here a sharper result for strong approximation proved by Leindler (see [8] and the literature cited there)

$$\max_{-\pi\leqslant x\leqslant\pi} \left(\frac{1}{n}\sum_{k=n+1}^{2n} |f(x)-s_k^*(f, x)|\right) \leqslant 7E_n(f). \tag{1.4}$$

Operators similar to these are also studied in the context of algebraic polynomial approximation, where, instead of the trigonometric Fourier series, one studies the Fourier series with respect to suitable orthogonal polynomials [2]. Such operators proved to be indispensable in the theory of weighted polynomial approximation (cf. [3–6, 14]). We observe that the operators $v_n^*$ are defined in terms of the Fourier coefficients of $f$, which in turn, involve the evaluation of integrals. In many applications (e.g., [15]), it is more desirable to have operators which have properties similar to (1.1), (1.2), and (1.3), but which are defined in terms of the values of the function $f$.

In the case of periodic functions, such quasi-interpolatory operators were defined by Bernstein [1] where also the boundedness result analogous to (1.3) is proved. In [20], Szabados proved similar results for certain operators based on the zeros of Chebyshev polynomials. In this paper, we generalize the results in two ways. First, we study operators based on the values of the function at the zeros of certain generalized Jacobi polynomials. Second, we study similar operators also in the case of certain Freud-type weight functions, supported on the whole real axis. In this paper, we have focused our attention on the $L^\infty$ behavior of the operators. The $L^p$ behavior is studied in [16].

In Section 2, we give the preliminary definitions and estimates. These are applied to the case of generalized Jacobi weights in Section 3 and the case of Freud-type weight functions in Section 4.

## 2. PRELIMINARIES

In this paper, for every real number $x \geqslant 0$, we denote the class of all algebraic polynomials of degree at most $x$ by $\Pi_x$. This somewhat unusual convention will actually simplify our notations later when we need to discuss polynomials of degree not exceeding $cn$ for some constant $c$ and integer $n$. Let $\alpha$ be a positive Borel measure on $\mathbb{R}$ having finite moments; i.e.,

$$\int |t|^r \, d\alpha(t) < \infty, \qquad r = 0, 1, 2, \ldots.$$

If $\alpha$ has at least $N$ points of increase, then there exists a unique system of polynomials

$$p_n(d\alpha; x) := \gamma_n(d\alpha)\, x^n + \cdots, \qquad \gamma_n(d\alpha) > 0, \quad n = 0, 1, \ldots, N-1,$$

such that

$$\int p_m(d\alpha; t)\, p_k(d\alpha; t)\, d\alpha(t) = \begin{cases} 1, & \text{if } \ k = m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

If $f$ is a Borel measurable function on $\mathbb{R}$, we write

$$a_k(d\alpha; f) := \int f(t)\, p_k(d\alpha; t)\, d\alpha(t), \qquad k = 0, 1, 2, \ldots, N-1,$$

whenever these integrals are well defined. The partial sum of the Fourier-orthonormal expansion of $f$ is then given by

$$s_m(d\alpha; f) := \sum_{k=0}^{m-1} a_k(d\alpha; f)\, p_k(d\alpha), \qquad m = 1, 2, \ldots, N.$$

Using (2.1), we obtain the integral representation

$$s_m(d\alpha; f, x) = \int f(t)\, K_m(d\alpha; x, t)\, d\alpha(t), \qquad m = 1, 2, \ldots, N,$$

where it is known [2] that the *Christoffel–Darboux kernel* $K_m$ can be expressed as

$$\begin{aligned} K_m(d\alpha; x, t) &:= \sum_{k=0}^{m-1} p_k(d\alpha; x)\, p_k(d\alpha; t) \\ &= \frac{\gamma_{m-1}(d\alpha)}{\gamma_m(d\alpha)} \frac{p_m(d\alpha; x)\, p_{m-1}(d\alpha; t) - p_m(d\alpha; t)\, p_{m-1}(d\alpha; x)}{x - t}. \end{aligned} \tag{2.2}$$

It is well known [2] that for each $n = 1, 2, ..., N-1$, the polynomial $p_n(d\alpha)$ has $n$ distinct zeros, all in the smallest interval $S(\alpha)$ containing the support of $d\alpha$. We denote these zeros by $x_{k,n}(d\alpha)$, with the ordering

$$x_{n,n}(d\alpha) < x_{n-1,n}(d\alpha) < \cdots < x_{1,n}(d\alpha).$$

If $\alpha$ has exactly $N$ points of increase, these points themselves will be denoted by $x_{k,N}$, $k = 1, ..., N$.

The *Cotes numbers* are defined by

$$\lambda_{k,n}(d\alpha) := \{K_n(d\alpha; x_{k,n}(d\alpha), x_{k,n}(d\alpha))\}^{-1}, \qquad k = 1, ..., n, \quad n = 1, 2, ..., N.$$

One of the most important properties of these is the (Gauss) *quadrature formula*:

$$\sum_{k=1}^{n} \lambda_{k,n}(d\alpha)\, P(x_{k,n}(d\alpha)) = \int P(t)\, d\alpha(t), \qquad P \in \Pi_{2n-1}, \quad n = 0, 1, ..., N. \tag{2.3}$$

In this paper, we sometimes find it convenient to express the left hand side of (2.3) as a Stieltjes integral. Thus, let $\alpha_n$ be the measure that associates the mass $\lambda_{k,n}(d\alpha)$ with $x_{k,n}(d\alpha)$, $k = 1, ..., n$. Then (2.3) can be written in the form

$$\int P(t)\, d\alpha_n(t) = \int P(t)\, d\alpha(t), \qquad P \in \Pi_{2n-1}, \quad n = 0, ..., N. \tag{2.4}$$

In the remainder of this section, we assume that $\alpha$ is a *mass distribution*; i.e., it is a positive Borel measure, all of whose moments are finite, and $\alpha$ has infinitely many points of increase. Thus, the orthonormal polynomials $p_k(d\alpha)$ are defined for all non-negative integers $k$.

We now proceed to define the operators which will be the discrete analogues of the de la Vallée Poussin means. For any integer $m \geqslant 1$, and Borel measurable function $f$ defined on the smallest interval containing the support of $\alpha$, we define the discrete Fourier coefficients of $f$ by

$$\begin{aligned} a_{k,m}(d\alpha; f) &:= \sum_{j=1}^{m} \lambda_{j,m}(d\alpha)\, f(x_{j,m}(d\alpha))\, p_k(d\alpha; x_{j,m}(d\alpha)) \\ &= \int f(t)\, p_k(d\alpha; t)\, d\alpha_m(t), \qquad k = 0, 1, .... \end{aligned}$$

The operators analogous to the de la Vallée Poussin operators are now defined by

$$\tau_{\ell,n,m}(d\alpha; f) := \sum_{k=0}^{\ell} a_{k,m}(d\alpha; f)\, p_k(d\alpha) + \sum_{k=\ell+1}^{2n-1} \left(\frac{2n-k}{2n-\ell}\right) a_{k,m}(d\alpha; f)\, p_k(d\alpha),$$
$$\ell = 0, ..., 2n-1; \quad n, m = 1, 2, ....$$

The cases $\ell = 0$, $\ell = n$, and $\ell = 2n-1$ can be seen as the discretized versions of the Fejér means, de la Vallée Poussin means, and Fourier sums, respectively. In particular, $\tau_{2n-1,n,2n}(d\alpha; f)$ is the classical Lagrange interpolation operator based on the zeros of $p_{2n}(d\alpha)$. Our main interest in this paper is in the cases when $\ell = 0$ and $\ell = n$. The proofs will also show the uniform boundedness of the operators (multiplied by suitable weight functions) in the case $\ell = [\kappa n]$ for $0 < \kappa < 2$. The results deteriorate quickly as $\kappa$ approaches 2.

The following Theorem 2.1 lists some basic properties of the operators $\tau_{\ell,n,m}$. In the sequel, if $f$ is a function defined on a borel set $A \subseteq \mathbb{R}$, we write

$$\|f\|_{\infty,A} := \sup_{t \in A} |f(t)|.$$

During the proof of Theorem 2.1, we will point out that when $m \geqslant 2n$,

$$\tau_{\ell,n,m}(d\alpha; f) = \frac{1}{2n-\ell} \sum_{k=\ell+1}^{2n} s_k(d\alpha_m; f).$$

Motivated by (1.4), we define the sublinear operator

$$\tau^{\#}_{\ell,n,m}(d\alpha; f) := \frac{1}{2n-\ell} \sum_{k=\ell+1}^{2n} |s_k(d\alpha_m; f)|. \tag{2.5}$$

THEOREM 2.1. *Let $\alpha$ be a mass distribution, $n \geqslant 1$, $0 \leqslant \ell \leqslant 2n-1$, $m \geqslant 2n$ be integers, and $Z_m$ be the set of zeros $\{x_{k,m}(d\alpha)\}_{k=1}^{m}$. Then*

$$\tau_{\ell,n,m}(d\alpha; P) = P, \qquad P \in \Pi_\ell.$$

*If $f: Z_m \to \mathbb{R}$, then $\tau_{\ell,n,m}(f) \in \Pi_{2n-1}$. Let $G: Z_m \to [0, \infty)$, $x \in S(\alpha)$, $I \subseteq S(\alpha)$ be an interval containing $x$, and $J$ be an interval such that $f(x_{k,m}(d\alpha)) = 0$ if $x_{k,m}(d\alpha) \in J$. Then the following estimate holds:*

$$\begin{aligned}
|\tau_{\ell,n,m}(d\alpha; f, x)| &\leqslant \tau^{\#}_{\ell,n,m}(d\alpha; f, x)\\
&\leqslant \|fG\|_{\infty, Z_m\setminus J} \sqrt{K_{2n}(d\alpha; x, x)}\\
&\quad\times \left\{ \sqrt{\int_{I\setminus J} \frac{1}{G^2(t)}\, d\alpha_m(t)} + \frac{2\Gamma_{2n}(d\alpha)}{2n-\ell} \right.\\
&\quad\times \left. \sqrt{\int_{S(\alpha)\setminus(I\cup J)} \frac{1}{G^2(t)(x-t)^2}\, d\alpha_m(t)} \right\},
\end{aligned} \tag{2.6}$$

*where*

$$\Gamma_{2n}(d\alpha) := \max_{1\leqslant j\leqslant 2n} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}.$$

*Proof.* Let $m \geqslant 2n$. If $j, k \leqslant 2n-1$ then $j+k \leqslant 4n-2 \leqslant 2m-1$, and the quadrature formula (2.4) yields

$$\int p_j(d\alpha; x)\, p_k(d\alpha; x)\, d\alpha_m(x) = \int p_j(d\alpha; x)\, p_k(d\alpha; x)\, d\alpha(x)$$

$$= \begin{cases} 1, & \text{if} \quad k=j,\\ 0, & \text{otherwise.} \end{cases}$$

In view of the uniqueness of orthogonal polynomial systems (cf. [2]), we obtain

$$p_k(d\alpha) = p_k(d\alpha_m), \qquad k=0, ..., 2n-1.$$

Therefore,

$$a_{k,m}(d\alpha; f) = a_k(d\alpha_m; f).$$

A simple computation then leads to

$$\tau_{\ell,n,m}(d\alpha; f) = \frac{1}{2n-\ell} \sum_{k=\ell+1}^{2n} s_k(d\alpha_m; f).$$

The operator $\tau_{\ell,n,m}$ is thus a discretization of the de la Vallée Poussin-type operator for the orthonormal polynomial expansions. The first inequality in (2.6) is now clear. Since $s_k(d\alpha_m; P) = P$ for every $P \in \Pi_\ell$, and $k = \ell+1, ..., 2n$, we obtain that $\tau_{\ell,n,m}(d\alpha; P) = P$ for all $P \in \Pi_\ell$. Also, it is clear that for any function $f$ defined on $Z_m$ we have $\tau_{\ell,n,m}(d\alpha; f) \in \Pi_{2n-1}$.

The estimate (2.6) is obtained using an argument similar to the one in [3, 18]. Let $I$ be a neighborhood of $x$. Then we define

$$A_x f(t) := \begin{cases} f(t), & \text{if} \quad t \in I, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_x f(t) := \begin{cases} 0, & \text{if} \quad t \in I, \\ \dfrac{f(t) - A_x f(t)}{x - t}, & \text{otherwise,} \end{cases}$$

to obtain

$$f(t) = A_x f(t) + (x - t)\, B_x f(t).$$

For $k \leqslant 2n$ we have

$$\begin{aligned} |s_k(d\alpha_m; A_x f, x)|^2 &= \left| \int_{I \setminus J} f(t)\, K_k(d\alpha; x, t)\, d\alpha_m(t) \right|^2 \\ &\leqslant \left\{ \int |K_k(d\alpha; x, t)|^2\, d\alpha_m(t) \right\} \left\{ \int_{I \setminus J} |f(t)|^2\, d\alpha_m(t) \right\} \\ &\leqslant K_k(d\alpha; x, x)\, \|f \cdot G\|^2_{\infty, Z_m \setminus J} \int_{I \setminus J} \frac{1}{G^2(t)}\, d\alpha_m(t). \end{aligned}$$

Hence,

$$\begin{aligned} &\tau^{\#}_{\ell, n, m}(d\alpha; A_x f, x) \\ &\quad \leqslant \sqrt{K_{2n}(d\alpha_m; x, x)} \sqrt{\int_{I \setminus J} \frac{1}{G^2(t)}\, d\alpha_m(t)}\, \|f \cdot G\|_{\infty, Z_m \setminus J}. \end{aligned} \tag{2.7}$$

Next, applying the Christoffel–Darboux-formula (2.2), we write

$$\begin{aligned} &s_k(d\alpha_m; f - A_x f, x) \\ &\quad = s_k(d\alpha_m; (x - \cdot\,)\, B_x f, x) \\ &\quad = \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} \int \frac{p_{k-1}(d\alpha; t)\, p_k(d\alpha; x) - p_{k-1}(d\alpha; x)\, p_k(d\alpha; t)}{x - t} \\ &\qquad \times (x - t)\, B_x f(t)\, d\alpha_m(t) \\ &\quad = \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} (p_k(d\alpha; x)\, a_{k-1}(d\alpha_m; B_x f) - p_{k-1}(d\alpha; x)\, a_k(d\alpha_m; B_x f)). \end{aligned}$$

Since

$$|s_k(d\alpha_m; f-A_xf, x)| \leqslant \Gamma_{2n}(d\alpha)(|p_k(d\alpha; x)\, a_{k-1}(d\alpha_m; B_xf)| + |p_{k-1}(d\alpha; x)\, a_k(d\alpha_m; B_xf)|),$$

we get using Bessel's inequality that

$$\begin{aligned}
\tau^{\#}_{\ell,n,m}&(d\alpha; f-A_xf, x)\\
&= \frac{1}{2n-\ell} \sum_{k=\ell+1}^{2n} |s_k(d\alpha_m; f-A_xf, x)|\\
&\leqslant \frac{2\Gamma_{2n}(d\alpha)}{2n-\ell} \sqrt{K_{2n}(d\alpha; x, x)} \sqrt{\sum_{k=\ell}^{2n} |a_k(d\alpha_m; B_xf)|^2}\\
&\leqslant \frac{2\Gamma_{2n}(d\alpha)}{2n-\ell} \sqrt{K_{2n}(d\alpha; x, x)} \sqrt{\int |B_xf(t)|^2\, d\alpha_m(t)}\\
&\leqslant \frac{2\Gamma_{2n}(d\alpha)}{2n-\ell} \sqrt{K_{2n}(d\alpha; x, x)} \sqrt{\int_{Z_m\setminus(I\cup J)} \frac{1}{G^2(t)(x-t)^2}\, d\alpha_m(t)}\\
&\quad \times \|f\cdot G\|_{\infty, Z_m\setminus(I\cup J)}.
\end{aligned} \tag{2.8}$$

Since

$$\tau^{\#}_{\ell,n,m}(d\alpha; f, x) \leqslant \tau^{\#}_{\ell,n,m}(d\alpha; A_xf, x) + \tau^{\#}_{\ell,n,m}(d\alpha; f-A_xf, x),$$

the second estimate in (2.6) is proved in view of (2.7) and (2.8). ∎

## 3. GENERALIZED JACOBI WEIGHTS

A *generalized Jacobi weight* is a function of the form

$$w(x) := \begin{cases} \prod_{k=1}^{\rho} |x-\xi_k|^{\beta_k}, & x \in [-1, 1],\\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\rho \geqslant 1$ is an integer, $-1 =: \xi_\rho < \cdots < \xi_1 := 1$, and $\beta_k > -1$ for $k = 1, ..., \rho$. The class of generalized Jacobi weights will be denoted by *GJ*; orthonormal polynomials with respect to a weight in *GJ* will be called *GJ* polynomials. These polynomials are studied extensively by Nevai in [19].

Following [19], if $w \in GJ$ is the form (3.1), and $m \geqslant 1$ is an integer, we write for $x \in [-1, 1]$,

$$\bar{w}_m(x) := \left(\sqrt{1-x} + \frac{1}{m}\right)^{2\beta_1+1} \prod_{k=2}^{\rho-1} \left(|x-\xi_k| + \frac{1}{m}\right)^{\beta_k} \left(\sqrt{1+x} + \frac{1}{m}\right)^{2\beta_\rho+1}.$$

If $w$ is the Legendre weight, $\beta_k = 0$, $k = 1, ..., \rho$, then it is easy to see that

$$\bar{w}_m(x) \sim \Delta_m(x) := \sqrt{1-x^2} + \frac{1}{m}.$$

In the sequel, we adopt the following convention regarding constants. The letters $c, c_1, ...$ will denote positive constants depending only on the weight function and other fixed parameters of the problem, but their value may be different in different occurences, even within the same formula. The notation $A \sim B$ denotes the fact that $cA \leqslant B \leqslant c_1 A$.

THEOREM 3.1. *Let $w \in GJ$, $d\alpha(x) := w(x)\, dx$, and*

$$G(x) := (1-x^2)^{\gamma/2} \sqrt{w(x)}, \qquad -1 < \gamma < 1.$$

*If $L \geqslant 2$, $n \geqslant 1$, $2n \leqslant m \leqslant Ln$ and $0 \leqslant \ell \leqslant 2n-1$ are integers, and $f\colon Z_m \to \mathbb{R}$, then*

$$\|\tau_{\ell,n,m}(d\alpha; f)\, \Delta^{\gamma}_{\sqrt{m}} \sqrt{\bar{w}_m}\|_{\infty,[-1,1]} \leqslant \|\tau^{\#}_{\ell,n,m}(d\alpha; f)\, \Delta^{\gamma}_{\sqrt{m}} \sqrt{\bar{w}_m}\|_{\infty,[-1,1]}$$
$$\leqslant \frac{cn}{2n-\ell} \|fG\|_{\infty, Z_m}. \tag{3.2}$$

*Proof.* We recall from [19] a few facts about the $GJ$ polynomials. Writing

$$x_{j,m}(d\alpha) =: \cos\theta_{j,m} =: x_{j,m}, \qquad j = 1, ..., m,$$

we have

$$0 < \theta_{j,m} < \pi, \qquad \theta_{j,m} - \theta_{j-1,m} \sim \frac{1}{m}, \qquad j = 1, ..., m, \tag{3.3}$$

where $\theta_{0,m} := 0$, $\theta_{m+1,m} := \pi$. Further, $\Gamma_n(d\alpha) \sim 1$,

$$K_{2n}(d\alpha; x, x) \sim \frac{n}{\bar{w}_n(x)}, \qquad x \in [-1, 1], \tag{3.4}$$

and

$$\lambda_{j,m} \sim \frac{1}{m} w(x_{j,m}) \sqrt{1-x_{j,m}^2}, \qquad j=1, ..., m. \tag{3.5}$$

Theorem 2.1 now implies that for $x \in [-1, 1]$, and any interval $I$ containing $x$,

$$\tau^{\#}_{\ell,n,m}(d\alpha; f, x) \leqslant c \, \|fG\|_{\infty, Z_m} \sqrt{\frac{n}{\bar{w}_n(x)}} \left\{ \sqrt{\int_I G^{-2}(t) \, d\alpha_m(t)} + \frac{1}{2n-\ell} \right.$$
$$\left. \times \sqrt{\int_{[-1,1]\setminus I} (G(t)(x-t))^{-2} \, d\alpha_m(t)} \right\}.$$

Using (3.5), we obtain for $x \in [-1, 1]$

$$\sqrt{\bar{w}_n(x)} \, \tau^{\#}_{\ell,n,m}(\alpha; f, x) \leqslant c \, \|fG\|_{\infty, Z_m} \left( \sqrt{S_1} + \frac{1}{2n-\ell} \sqrt{S_2} \right), \tag{3.6}$$

where, with $\delta := 1-2\gamma$,

$$\begin{aligned} S_1 &:= \sum_{x_{j,m} \in I} (1-x_{j,m}^2)^{\delta/2}, \\ S_2 &:= \sum_{x_{j,m} \in [-1,1]\setminus I} \frac{(1-x_{j,m}^2)^{\delta/2}}{(x-x_{j,m})^2}. \end{aligned} \tag{3.7}$$

We will estimate $S_1$ and $S_2$ for $x \geqslant 0$; the case when $x<0$ is similar. In the remainder of the proof, $x =: \cos\phi$ is a fixed number, with $0 \leqslant \phi \leqslant \pi/2$.

*Case* 1. $(0 \leqslant \phi \leqslant 2/\sqrt{m})$.

We write $I^\circ := [0, \phi + 2/\sqrt{m}]$, and

$$I := \{x = \cos\theta \colon \theta \in I^\circ\}. \tag{3.8}$$

In view of (3.3) and the fact that $\delta > -1$,

$$\begin{aligned} \frac{1}{m} S_1 &= \frac{1}{m} \sum_{\theta_{j,m} \in I^\circ} \sin^\delta \theta_{j,m} \leqslant \frac{c}{m} \sum_{\theta_{j,m} \in I^\circ} \theta_{j,m}^\delta \\ &\leqslant c \int_0^{\phi + 3/\sqrt{m}} t^\delta \, dt \leqslant c \left( \phi + \frac{1}{\sqrt{m}} \right)^{\delta+1}. \end{aligned} \tag{3.9}$$

Since $\phi + 1/\sqrt{m} \sim 1/\sqrt{m}$, we deduce that

$$S_1 \leqslant c\left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 1} \leqslant c\left(\sin\phi + \frac{1}{\sqrt{m}}\right)^{-2\gamma} = c\Delta_{\frac{1}{\sqrt{m}}}^{-2\gamma}(x). \tag{3.10}$$

It is easy to check that $\sin\theta \sim \theta$ if $\theta \in [0, 3\pi/4]$, and hence, that

$$S_2 \leqslant c \sum_{\theta_{j,m} \in [0,\pi] \setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2}. \tag{3.11}$$

If $\theta_{j,m} \in [\pi/2, \pi]$, then $\theta_{j,m}^2 - \phi^2 \geqslant c$. Using (3.3), and the fact that $\delta > -1$, we obtain

$$\begin{aligned} \sum_{\theta_{j,m} \in [\pi/2,\pi]} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} &\leqslant c \sum_{\theta_{j,m} \in [\pi/2,\pi]} \sin^\delta(\pi - \theta_{j,m}) \\ &\leqslant cm \int_0^{3\pi/4} t^\delta \, dt \leqslant cm. \end{aligned}$$

Since $\delta < 3$ and $\phi + 1/\sqrt{m} \sim 1/\sqrt{m}$,

$$m \leqslant m^2 m^{-(\delta - 1)/2} \leqslant cm^2 \left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 1}.$$

Hence,

$$\sum_{\theta_{j,m} \in [\pi/2,\pi]} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} \leqslant cm^2 \left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 1}. \tag{3.12}$$

If $0 \leqslant 2\phi \leqslant \phi + 2/\sqrt{m} \leqslant \theta_{j,m} \leqslant \pi/2$, then $\theta_{j,m}^2 - \phi^2 \geqslant c\theta_{j,m}^2$ and $\sin\theta_{j,m} \sim \theta_{j,m}$. Hence, using (3.3) and the fact that $\delta < 3$, we get

$$\begin{aligned} \sum_{\theta_{j,m} \in [0,\pi/2] \setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} &\leqslant cm \left\{ \frac{1}{m} \sum_{\theta_{j,m} \in [0,\pi/2] \setminus I^\circ} \theta_{j,m}^{\delta - 4} \right\} \\ &\leqslant cm \int_{\phi + 1/\sqrt{m}}^{\infty} t^{\delta - 4} \, dt \\ &\leqslant cm \left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 3} \\ &\leqslant cm^2 \left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 1}. \end{aligned}$$

Along with (3.11) and (3.12), this gives

$$S_2 \leqslant cm^2 \left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 1} \leqslant cm^2 \Delta^{-2\gamma}_{\sqrt{m}}(x). \tag{3.13}$$

The estimates (3.6), (3.10), (3.13) yield that when $0 \leqslant \phi \leqslant 2/\sqrt{m}$, we have

$$\begin{aligned}\Delta^{\gamma}_{\sqrt{m}}(x)\sqrt{\bar{w}_n(x)}\,\tau^{\#}_{\ell, n, m}(d\alpha; f, x) &\leqslant c\left(1 + \frac{m}{2n-\ell}\right)\|fG\|_{\infty, Z_m}\\ &\leqslant \frac{cn}{2n-\ell}\|fG\|_{\infty, Z_m}.\end{aligned} \tag{3.14}$$

*Case* 2. $(2/\sqrt{m} < \phi \leqslant \pi/2)$.

In this case, we take $I^\circ := [\phi - 1/(m\phi), \phi + 1/(m\phi)]$, and $I := \{\cos\theta : \theta \in I^\circ\}$. The following estimates will be used in the remainder of this proof often, sometimes without an explicit reference:

$$\begin{aligned}\frac{1}{4}\left(\phi + \frac{1}{\sqrt{m}}\right) \leqslant \frac{1}{2}\left(\phi + \frac{1}{m\phi}\right) \leqslant \phi - \frac{1}{m\phi}\\ \leqslant \phi \leqslant \phi + \frac{1}{m\phi} \leqslant \phi + \frac{1}{\sqrt{m}}.\end{aligned} \tag{3.15}$$

Also, in view of (3.3), the number of $\theta_{j,m}$'s in $I^\circ$ is at most $c/\phi \leqslant c(\phi + 1/\sqrt{m})^{-1}$. Hence,

$$S_1 \leqslant c \sum_{\theta_{j,m} \in I^\circ} \theta^{\delta}_{j,m} \leqslant c\left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta - 1} \leqslant c\Delta^{-2\gamma}_{\sqrt{m}}(x). \tag{3.16}$$

As in Case 1, we deduce easily that

$$\sum_{\theta_{j,m} \in [3\pi/4, \pi]} \frac{\sin^{\delta}\theta_{j,m}}{(\theta^2_{j,m} - \phi^2)^2} \leqslant cm.$$

If $1 \leqslant \delta < 3$ then

$$\frac{1}{m} \leqslant m^{-(\delta - 1)/2} \leqslant c\phi^{\delta - 1},$$

and if $-1<\delta<1$ then it is clear that $1/m \leqslant c\phi^{\delta-1}$. Thus, in either case, $m \leqslant m^2(\phi+1/\sqrt{m})^{\delta-1}$, and we get

$$\sum_{\theta_{j,m}\in[3\pi/4,\pi]} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2-\phi^2)^2} \leqslant cm^2 \Delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.17}$$

Next, we write

$$I_{2,1} := \left[0, \frac{1}{2}\left(\phi-\frac{1}{m\phi}\right)\right],$$

$$I_{2,2} := \left[\frac{1}{2}\left(\phi-\frac{1}{m\phi}\right), \phi-\frac{1}{m\phi}\right],$$

$$I_{2,3} := \left[\phi+\frac{1}{m\phi}, 2\left(\phi+\frac{1}{m\phi}\right)\right],$$

$$I_{2,4} := \left[2\left(\phi+\frac{1}{m\phi}\right), 3\pi/4\right],$$

$$S_{2,j} := \sum_{\theta_{j,m}\in I_{2,j}} \frac{\theta_{j,m}^\delta}{(\theta_{j,m}^2-\phi^2)^2}, \qquad j=1,2,3,4.$$

Then

$$\sum_{\theta_{j,m}\in[0,3\pi/4]\setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2-\phi^2)^2} \leqslant c \sum_{j=1}^{4} S_{2,j}. \tag{3.18}$$

We estimate $S_{2,3}$ and $S_{2,4}$; the estimates for $S_{2,2}$ and $S_{2,1}$ are similar. Using (3.3),

$$\begin{aligned} S_{2,3} &\leqslant \frac{c}{\phi^2}\left(\phi+\frac{1}{m\phi}\right)^\delta \sum_{\theta_{j,m}\geqslant \phi+1/(m\phi)} \frac{1}{(\theta_{j,m}-\phi)^2} \\ &\leqslant c\phi^{\delta-2} m \int_{\phi+1/(m\phi)}^{\infty} (t-\phi)^{-2}\,dt \\ &\leqslant cm^2\phi^{\delta-1} \leqslant cm^2\left(\phi+\frac{1}{\sqrt{m}}\right)^{\delta-1}. \end{aligned} \tag{3.19}$$

If $\theta \in I_{2,4}$, then $\theta-\phi \geqslant \theta/2$. Using (3.3) and the fact that $\delta<3$, we get

$$\begin{aligned} S_{2,4} &\leqslant c \sum_{\theta_{j,m}\in I_{2,4}} \theta_{j,m}^{\delta-4} \leqslant cm \int_{\phi+1/(m\phi)}^{\infty} t^{\delta-4}\,dt \\ &\leqslant cm\left(\phi+\frac{1}{m\phi}\right)^{\delta-3} \leqslant cm^2\left(\phi+\frac{1}{\sqrt{m}}\right)^{\delta-1}. \end{aligned} \tag{3.20}$$

From (3.18), (3.19), (3.20), and similar estimates for $S_{2,1}$ and $S_{2,2}$, we obtain

$$\sum_{\theta_{j,m}\in[0,3\pi/4]\setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2-\phi^2)^2} \leqslant cm^2\left(\phi+\frac{1}{\sqrt{m}}\right)^{\delta-1} \leqslant cm^2 \Delta_{\sqrt{m}}^{-2\gamma}(x).$$

Along with (3.17), this yields

$$S_2 \leqslant cm^2 \delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.21}$$

From (3.6), (3.16), and (3.21), we conclude that

$$\begin{aligned}\Delta_{\sqrt{m}}^{\gamma}(x)\sqrt{\bar{w}_n(x)}\,\tau_{\ell,n,m}^{\#}(d\alpha; f, x) &\leqslant c\left(1+\frac{m}{2n-\ell}\right)\|fG\|_{\infty, Z_m}\\ &\leqslant \frac{cn}{2n-\ell}\|fG\|_{\infty, Z_m}.\end{aligned} \tag{3.22}$$

The estimates (3.14), (3.22), and analogous estimates for $x\in[-1,0)$ yield (3.2). ∎

We end this section by observing the "continuous analogue" of Theorem 3.1. It is probably not new, but we are unable to locate a precise reference. The proof of the following theorem is similar to that of Theorem 3.1, but simpler.

THEOREM 3.2. *Let $w\in GJ$, $d\alpha(x) := w(x)\,dx$, and $G$, $L$, $m$, $n$, $\ell$ be as in Theorem 3.1. Let $f\colon[-1,1]\to\mathbb{R}$ be a measurable function such that $fG$ is essentially bounded on $[-1,1]$. Then for $x\in[-1,1]$,*

$$\Delta_{\sqrt{m}}^{\gamma}(x)\sqrt{\bar{w}_m(x)}\,\frac{1}{n}\sum_{k=\ell+1}^{2n}|s_k(d\alpha; f, x)| \leqslant \frac{cn}{2n-\ell}\|fG\|_{\infty,[-1,1]}. \tag{3.23}$$

In [16], we have examined the $L^p$ versions of Theorems 3.1 and 3.2 (with $\ell = n$) using certain analogues of the so called Marcinkiewicz–Zygmund type inequalities. In the case of the weights in GJ, such inequalities have been studied by Mastroianni, Totik, Vértesi, and Xu [23], [24], [11], [12], [13], among others.

## 4. FREUD-TYPE WEIGHT FUNCTIONS

Let $w\colon \mathbb{R}\to(0,\infty)$, and $Q := \log(1/w)$. The function $w$ is called a Freud-type weight function if each of the following conditions is satisfied. The function $Q$ is an even, convex function on $\mathbb{R}$, $Q$ is twice continuously differentiable on $(0, \infty)$, and there are constants $c_1$ and $c_2$ such that

$$0 < c_1 \leqslant \frac{xQ''(x)}{Q'(x)} \leqslant c_2 < \infty, \qquad 0 < x < \infty. \tag{4.1}$$

The most commonly discussed examples include $\exp(-|x|^\alpha)$, $\alpha > 1$. In the remainder of this section, $w$ will denote a fixed Freud-type weight function, $\{p_k\}$ will denote the sequence of polynomials orthonormal on $\mathbb{R}$ with respect to the measure $w^2(x)\,dx$. From all notations, we will omit the mention of this measure; thus $x_{k,n}$ will be the $k$th zero of $p_n$, etc.

Our main theorem in this section is the following.

THEOREM 4.1. *Let $w$ be a Freud-type weight function. If $f\colon Z_m \to \mathbb{R}$, and $L, \delta > 0$, then for each integer $n \geqslant 1$ and $(2+\delta)\,n \leqslant m \leqslant Ln$,*

$$\|\tau_{\ell,n,m}(f)\,w\|_{\infty,\mathbb{R}} \leqslant \|\tau^{\#}_{\ell,n,m}(f)\,w\|_{\infty,\mathbb{R}} \leqslant \frac{cn}{2n-\ell}\,\|wf\|_{\infty,Z_m}, \tag{4.2}$$

*where $Z_m = \{x_{k,m}\}_{k=1}^{m}$, and $c$ is a positive constant depending only on $w$, $L$, and $\delta$.*

In order to prove this theorem, we summarize some of the important, relevant facts regarding Freud polynomials. Associated with the weight function $w$ are two sets of numbers: The Freud-number $q_x$ is the least positive solution of the equation

$$q_x Q'(q_x) = x, \qquad x > 0.$$

The number $a_x$ is the solution of the equation

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x t Q'(a_x t)}{\sqrt{1-t^2}}\,dt.$$

It is not difficult to see that $a_x \sim q_x \sim q_{2x}$, $x > 0$. One of the most important properties of $a_x$ is the following: For every integer $n \geqslant 1$ and $P \in \Pi_n$, (cf. [17], [14]),

$$\max_{x \in \mathbb{R}} |P(x)\,w(x)| = \max_{|x| \leqslant a_n} |P(x)\,w(x)|,$$

and, if $0 < p < \infty$, $N$ is the least integer not exceeding $n + 2/p$, then

$$\int_{\mathbb{R}} |P(x)\,w(x)|^2 \leqslant 2 \int_{|x| \leqslant a_N} |P(x)\,w(x)|^p\,dx. \tag{4.3}$$

LEMMA 4.1. *Let $n \geqslant 1$ be an integer.*

(a) *For $x \in \mathbb{R}$,*

$$K_n(x, x) \leqslant c \frac{n}{q_n} w^{-2}(x). \tag{4.4.}$$

(b) *We have*

$$\Gamma_n \sim q_n. \tag{4.5}$$

(c) *For any $\eta > 0$ and $|x| \leqslant (1-\eta)\, a_n$,*

$$K_n(x, x) \geqslant c(\eta) \frac{n}{q_n} w^{-2}(x). \tag{4.6}$$

(d) *For any $\eta > 0$ and $x_{k,n}$, $x_{k-1,n}$, $x_{k+1,n}$ in $[-(1-\eta)\, a_n, (1-\eta)\, a_n]$,*

$$x_{k-1,n} - x_{k,n} \sim x_{k,n} - x_{k+1,n} \sim \frac{q_n}{n}.$$

(e) *If $|x-t| \leqslant cq_n/n$, $|x|, |t| \leqslant c_1 q_n$, then $w(x) \sim w(t)$.*

(f) *Let $b \in \mathbb{R}$ and $0 \leqslant p \leqslant 2$. Then*

$$\begin{aligned} c \int_0^{c_1 q_n} w^{2-p}(x)(1+x^2)^b \, dx &\leqslant \sum_{j=1}^{n} \lambda_{j,n} w^{-p}(x_{j,n})(1+x_{j,n}^2)^b \\ &\leqslant c_2 \int_0^{c_3 q_n} w^{2-p}(x)(1+x^2)^b \, dx. \end{aligned} \tag{4.7}$$

*Proof.* Part (a) and (b) are proved in [6]. Parts (c) and (d) are in the paper [9] by Levin and Lubinsky. The most difficult part of the proof is a judicious discretization of a certain logarithmic potential. A simpler construction is given in [14] (cf. [22]) under slightly stronger conditions, which are also satisfied by $\exp(-|x|^\alpha)$, $\alpha > 1$. Using (4.1), it is not difficult to verify (cf. [6]) that $Q'(Aq_n) \sim n/q_n$ for any $A > 0$. Part (e) is then a simple application of the mean value theorem. Part (f) was proved by Knopmacher and Lubinsky [7] under slightly different conditions on the weight function. As stated, the result is proved in [14] (Theorem 8.2.7) using their ideas. ∎

We are now in a position to prove Theorem 4.1.

*Proof of Theorem* 4.1. In this proof, let $x \in [-a_{2n}, a_{2n}]$ be a fixed number, and $v$ be an integer such that $x \in [x_{v+1,m}, x_{v,m})$. The constants in this proof will generally depend upon $\delta$ and $L$. Necessarily, there exists $\eta > 0$ such that $|x| \leqslant (1-\eta)\, a_m$. We use the estimate (2.6) with $I = [x - q_m/m, x + q_m/m]$, with $J$ equal to the empty interval, and $G = w$. The set $S(\alpha)$ in this case is $\mathbb{R}$. In view of (4.4) and (4.5), we obtain the estimate

$$|w(x)\, \tau_{\ell,n,m}(f, x)| \leqslant c\, \|fw\|_{\infty, z_m} \left\{ \sqrt{\frac{n}{q_n} \int_I \frac{1}{w^2(t)}\, d\alpha_m(t)} + \sqrt{\frac{nq_n}{(2n-\ell)^2} \int_{\mathbb{R}\setminus I} \frac{1}{w^2(t)(x-t)^2}\, d\alpha_m(t)} \right\}. \tag{4.8}$$

The number of points $x_{j,m}$ in $I$ is bounded from above, independently of $n$ and $m$. For each such $x_{j,m} \in I$, (4.6) (with $m$ in place of $n$ and a different value of $\eta$) shows that

$$\lambda_{j,m} w^{-2}(x_{j,m}) \leqslant c\, \frac{q_m}{m}.$$

Hence,

$$\int_I w^{-2}(t)\, d\alpha_m(t) \leqslant c\, \frac{q_m}{m} \leqslant c\, \frac{q_n}{n}. \tag{4.9}$$

Let

$$I_1 := \left[ -\left(1 - \frac{\eta}{2}\right) a_m, \left(1 + \frac{\eta}{2}\right) a_m \right] \Big\backslash I$$

and

$$I_2 := \left( -\infty, -\left(1 - \frac{\eta}{2}\right) a_m \right) \cup \left( \left(1 - \frac{\eta}{2}\right) a_m, \infty \right).$$

In view of (4.7) with $b = 0$ and $p = 2$,

$$\sum_{x_{j,m} \in I_2} \frac{w^{-2}(x_{j,m})\, \lambda_{j,m}}{(x - x_{j,m})^2} \leqslant c a_m^{-2} \sum_{j=1}^{n} w^{-2}(x_{j,m})\, \lambda_{j,m} \leqslant c q_m^{-1}. \tag{4.10}$$

If $x_{j,m} \in I_1$, then

$$|x - x_{j,m}| \sim |x - x_{j-1,m}| \sim |x - x_{j+1,m}|$$

and

$$w^{-2}(x_{j,m})\,\lambda_{j,m} \sim \frac{q_m}{m} \sim x_{j,m} - x_{j+1,m}.$$

Therefore, taking into account the fact that $m \sim n$, we obtain

$$\sum_{x_{j,m}\in I_1} \frac{w^{-2}(x_{j,m})\,\lambda_{j,m}}{(x-x_{j,m})^2} \leqslant \sum_{x_{j,m}\in I_1} \frac{x_{j,m}-x_{j+1,m}}{(x-x_{j,m})^2}$$
$$\leqslant \int_{\mathbb{R}\setminus I} \frac{dt}{(x-t)^2} \leqslant c\,\frac{m}{q_m} \leqslant c\,\frac{n}{q_n}. \tag{4.11}$$

Substituting the estimates (4.11), (4.10) and (4.9) into (4.8), we arrive at (4.2). ∎

We end this section by observing the following amusing inequality, obtained by using Theorem 4.1. with a polynomial of degree $n$ in place of $f$, and observing that $\tau_{n,n,m}(P) = P$ for all $P \in \Pi_n$.

COROLLARY 4.1. *Let $w$, $L$, $\delta$, $n$, $m$, $Z_m$ be as in Theorem* 4.1. *Then for any $P \in \Pi_n$,*

$$c_1 \|wP\|_{\infty,\mathbb{R}} \leqslant \|wP\|_{\infty,Z_m} \leqslant \|wP\|_{\infty,\mathbb{R}}. \tag{4.12}$$

In [16], we have shown how estimate such as (4.12) together with the uniform boundedness of the de la Vallée Poussin means of the Freud polynomial expansions lead to estimates similar to (4.12) with an $L^p$ norm instead of the supremum norm. In turn, it is well known that such estimates are important in the study of interpolatory processes at the zeros of Freud polynomials.

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